Some useful results concerning the representation theory of the symmetric group



1991 J. Phys. A: Math. Gen. 24 5227

(http://iopscience.iop.org/0305-4470/24/22/006)

J Phys. A Math Gen 24 (1991) 5227–5234. Printed in the UK

# Some useful results concerning the representation theory of the symmetric group

J Katriel

Department of Chemistry, Technion–Israel Institute of Technology, 32000 Haifa, Israel

Received 17 December 1990, in final form 9 May 1991

**Abstract.** A theorem concerning the explicit form of the eigenvalues of the class sums of the symmetric group ($S_n$) is derived and used to obtain the following results. (1) the centre of the $S_n$-algebra is generated by means of polynomials in the set of elements consisting of the generators of the centre of the $S_{n-k}$ algebra augmented by the single cycle class sums $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$. (2) the irreps of $S_n$ with up to $k$ rows are fully specified by the class sums $[(2)]_n, [(3)]_n, \ldots, [(k)]_n$. Furthermore, it is found that the $k$ class sums $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$ suffice to specify the irreps of $S_n$ for all $n \leqslant n_{\max}(k)$, where $n_{\max}(k) \gg k$.

## 1. Introduction

The symmetric group occupies a central position in the theory of finite as well as unitary groups. As a consequence of the indistinguishability of identical elementary particles in quantum mechanics, the symmetric group has to be applied in order to specify the various states of any many-body system. This is an essential step in all procedures for the study of the spectrum and dynamics of the corresponding Hamiltonian.

The representation theory of the symmetric group has been investigated very thoroughly and extensively since the turn of the century, or even earlier. The classical results are fully described in [1–4]. It may therefore appear surprising that interesting open problems still exist, and even more surprising that simple and useful new results can still be obtained using elementary ideas. The point of view followed in the present paper was very effectively advocated by Chen [5], who emphasized the role of the set of elements spanning the centre of the group algebra. These elements, known as the class sums, form a complete set of mutually commuting operators, whose common eigenfunctions belong to well-defined irreducible representations (irreps). The corresponding eigenvalues, denoted $\lambda_C^\Gamma$, are related to the characters $\chi_C^\Gamma$ by means of

$$\lambda_C^\Gamma = \frac{|C|}{|\Gamma|} \chi_C^\Gamma$$

where $|\Gamma|$ and $|C|$ stand for the degeneracy of the irrep $\Gamma$ and the number of elements in the class $C$, respectively. The present author has recently been able to make considerable progress towards a combinatorial theory of the structure constants in the group algebra of the symmetric group [6–8].

In this paper a theorem concerning the form of the expressions for the eigenvalues of the single-cycle class sums of the symmetric group is derived and used to obtain the following results. (1) the centre of the $S_n$-algebra is generated by means of polynomials in the set of elements consisting of the generators of the centre of the $S_{n-k}$ algebra augmented by the single-cycle class sums $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$; (2) the irreps of $S_n$ with up to $k$ rows are fully specified by the class sums $[(2)]_n$, $[(3)]_n, \ldots, [(k)]_n$. It is found that the $k$ class sums $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$ suffice to specify the irreps of $S_n$ for all $n \leqslant n_{\max}(k)$, where $n_{\max}(k) \gg k$. The relevance of these results to the construction of multi-cluster wavefunctions and to the treatment of spin Hamiltonians with arbitrary elementary spins is briefly pointed out in the appropriate sections below.

## 2. Preliminary results

In this section we introduce some basic notions and formulate three lemmas which are used in the following sections.

Given a set of $k$ variables $\{x_1, x_2, \ldots, x_k\}$ we define a set of $k$ power sums

$$p_r = \sum_{i=1}^{k} x_i^r \qquad r = 1, 2, \ldots, k. \tag{1}$$

These power sums determine the $k$ elementary symmetric functions [2]

$$e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} \qquad r = 1, 2, \ldots, k \tag{2}$$

which are the coefficients in the polynomial

$$\prod_{i=1}^{k} (x - x_i) = x^k - x^{k-1} e_1 + x^{k-2} e_2 \ldots = \sum_{r=0}^{k} (-1)^r x^{k-r} e_r \tag{3}$$

where $e_0 = 1$. From the fundamental theorem of algebra the following applies.

*Lemma 1.* The set of $k$ power sums $\{p_1, p_2, \ldots, p_k\}$ determines the set of variables $\{x_1, x_2, \ldots, x_k\}$ up to permutations among them.

A Young shape consisting of $n$ boxes is usually specified in terms of the lengths of its rows, which form a partition of $n$ ($\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$, where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$). Denoting each box by a row index $i$ (running from top to bottom) and a column index $j$ (running from left to right) we follow Robinson and Thrall [1, 9] and refer to the difference $-ij$ as the content of the box $(i, j)$. A Young shape specifies a set of $n$ contents. It is shown in [1, 9] that any set of contents specifies at most one Young shape.

Recalling that an irrep of the symmetric group is fully specified by a corresponding Young shape we obtain the following.

*Lemma 2.* Any irrep of the symmetric group $S_n$ is specified by an appropriate set of $n$ contents.

Combining lemma 1 and lemma 2 we conclude the following.

*Lemma 3.* Any irrep of the symmetric group is specified by a set of $n$ power sums over a set of $n$ contents.

The power sums over the contents of a Young shape Γ will be denoted by

$$\sigma_r = \sum_{(i,j)\in\Gamma} (-ij)^r \qquad r=1,2,\ \ldots \tag{4}$$

## 3. On the eigenvalues of the class sums of the symmetric group

Each class sum of the symmetric group [5] possesses a set of eigenvalues corresponding to the irreps of this group. Expressions for the eigenvalues of the class sums $[(2)]_n$, $[(3)]_n$ and $[(4)]_n$ in terms of the lengths of the different rows of the Young shape specifying the irrep were given by Partensky [10]. It follows from lemma 3 that the eigenvalues of any class sum can be expressed in terms of the power sums over the contents of the corresponding Young shapes. Expressions of this form for the eigenvalues for the classes of the transpositions and the 3-cycles

$$\lambda^{\Gamma}_{[(2)]_n} = \sigma_1 \tag{5}$$

$$\lambda^{\Gamma}_{[(3)]_n} = \sigma_2 - \tfrac{1}{2}n(n-1) \tag{6}$$

were given by Jucys [11] and, independently, by Suzuki [12]. These expressions are equivalent to those due to Partensky [10]. A heuristic construction of the expressions for the eigenvalues of the single-cycle class sums with up to 14 indices was recently proposed by Pauncz and Katriel [13]. Of these, the eigenvalues of the 4- and 5-cycles are

$$\lambda^{\Gamma}_{[(4)]_n} = \sigma_3 - (2n-3)\sigma_1 \tag{7}$$

$$\lambda^{\Gamma}_{[(5)]_n} = \sigma_4 - (3n-10)\sigma_2 - 2\sigma_1^2 + \tfrac{1}{6}n(n-1)(5n-19). \tag{8}$$

These results suggest the following theorem.

*Theorem 1.* The eigenvalues of the class sums containing a single cycle of length $p$, in an irrep Γ, are polynomials in the $p-1$ power sums $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$, with coefficients which are polynomials in $n$

This theorem is proved by considering the expression for the eigenvalue of the class sum $[(1)^{n-p}(p)]_n$ in the irrep Γ specified by the partition $\lambda_1+\lambda_2+\ldots+\lambda_k = n$ ($\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$), which was presented by MacDonald [2], following Frobenius. This expression involves the $n$ variables

$$\mu_i = \lambda_i - i + n \qquad i=1,2,\ldots,n \tag{9}$$

which satisfy

$$\mu_1 > \mu_2 > \ldots > \mu_n \tag{10}$$

i.e., they are strictly decreasing. It can be written in the form

$$\lambda^{\Gamma}_{[(p)]_n} = \frac{1}{p}\sum_{i=1}^{n} \mu_i(\mu_i - 1)\ldots(\mu_i - p + 1)\prod_{j\neq i}\frac{(\mu_i - \mu_j - p)}{(\mu_i - \mu_j)}. \tag{11}$$

This is a rational symmetric function in the $n$ variables $\mu_1, \mu_2, \ldots, \mu_n$. We shall now show that viewing $\lambda^{\Gamma}_{[(p)]_n}$ as a rational function in any one variable, keeping all the others constant, it has no poles. This implies that this rational function is actually a polynomial of degree $p$ in each one of the variables. Being symmetric in the $n$ variables it can be expressed in terms of the symmetric power sums

$$\tau_k = \sum_{i=1}^{n} \mu_i^k \qquad k = 1, 2, \ldots, p. \tag{12}$$

These power sums can be written in terms of the first $p-1$ power sums in the contents of the corresponding Young shape. Therefore, the expression for the eigenvalues of the class sum $[(p)]_n$, presented in equation (11), can be expressed in terms of the latter.

To show the absence of poles in $\lambda^{\Gamma}_{[(p)]_n}$ we choose $\mu_1$ as the variable to be examined. Denoting it by $x$ we obtain

$$\lambda(x) = \frac{1}{p} x(x-1)\ldots(x-p+1) \prod_{j=2}^{n} \left(1 - \frac{p}{x-\mu_j}\right) + \frac{1}{p} \sum_{i=2}^{n} \mu_i(\mu_i - 1)\ldots(\mu_i - p + 1)\left(1 - \frac{p}{\mu_i - x}\right) \prod_{\substack{j=2 \\ (j \neq i)}}^{n} \left(1 - \frac{p}{\mu_i - \mu_j}\right). \tag{13}$$

The only values of $x$ for which $\lambda(x)$ could have poles are $\mu_2, \mu_3, \ldots, \mu_n$. To investigate the behaviour of $\lambda(x)$ in the neighbourhood of $\mu_k$ we separate the regular part and write

$$\lambda(x) = \lambda_{\mathrm{R}}(x) + \frac{1}{\mu_k - x}\left[x(x-1)\ldots(x-p+1) \prod_{\substack{j=2 \\ (j \neq k)}}^{n} \left(1 - \frac{p}{x-\mu_j}\right) - \mu_k(\mu_k - 1)\ldots(\mu_k - p + 1) \prod_{\substack{j=2 \\ (j \neq k)}}^{n} \left(1 - \frac{p}{\mu_k - \mu_j}\right)\right] \tag{14}$$

where

$$\lambda_{\mathrm{R}}(x) = \frac{1}{p} x(x-1)\ldots(x-p+1) \prod_{\substack{j=2 \\ (j \neq k)}}^{n} \left(1 - \frac{p}{x-\mu_j}\right) + \frac{1}{p} \sum_{\substack{i=2 \\ (i \neq k)}}^{n} \mu_i(\mu_i - 1)\ldots(\mu_i - p + 1)\left(1 - \frac{p}{\mu_i - x}\right) \prod_{\substack{j=2 \\ (j \neq i)}}^{n} \left(1 - \frac{p}{\mu_i - \mu_j}\right) + \frac{1}{p}\mu_k(\mu_k - 1)\ldots(\mu_k - p + 1) \prod_{\substack{j=2 \\ (j \neq k)}}^{n} \left(1 - \frac{p}{\mu_k - \mu_j}\right). \tag{15}$$

Using l'Hospital's rule we obtain

$$\lim_{x \to \mu_k} [\lambda(x) - \lambda_{\mathrm{R}}(x)] = -\frac{\mathrm{d}}{\mathrm{d}x}\left[x(x-1)\ldots(x-p+1) \prod_{\substack{j=2 \\ (j \neq k)}}^{n} \left(1 - \frac{p}{x-\mu_j}\right)\right]\Bigg|_{x=\mu_k} \tag{16}$$

which is finite (by equation (10)). This completes the proof of theorem 1.

To explicate the relation between the symmetric power sums $\tau_k$ defined in equation (12) and the power sums over the contents of the Young shape we note that the contents in the $i$th row are $-i+1, -i+2, \ldots, 0, 1, 2, \ldots, \lambda_i - i$. The contribution of this row to $\sigma_k$ is

$$\sum_{j=1}^{\lambda_i - i} j^k + (-1)^k \sum_{j=1}^{i-1} j^k. \tag{17}$$

Since the first sum can be written as a polynomial of degree $k+1$ in $(\lambda_i - i)$, it follows that $\sigma_k$ can be expressed linearly in terms of $\tau_2, \ldots, \tau_{k+1}$. (Note that $\tau_1 = \frac{1}{2}n(n+1)$.)

By a simple inductive argument we obtain from theorem 1 the following.

*Lemma 4.* The eigenvalues of $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$ $(k<n)$ determine the power sums $\sigma_1(n), \sigma_2(n), \ldots, \sigma_k(n)$.

We note in passing that the set of power sums $\sigma_1(n), \sigma_2(n), \ldots, \sigma_{n-1}(n)$ determines the power sums $\sigma_l(n)$ with $l \geq n$.

## 4. Generation of the centre of the symmetric group algebra

The centre of the symmetric group algebra $(CS_n)$ is linearly generated by the class sums of its various classes. Several results have been obtained by various authors concerning a smaller set of generators. Farahat and Higman [14] classified the various classes according to the number of cycles and showed that the set of sums of the form

$$\left( \sum_{\substack{l_1, l_2, \ldots, l_n \\ (\Sigma_i i l_i = n, \Sigma_i l_i = L)}} [(1)^{l_1}(2)^{l_2} \ldots (n)^{l_n}]_n ; L = 1, 2, \ldots, n \right) \tag{18}$$

generates the centre of the group algebra, i.e. each class sum can be expressed as a polynomial in these $n$ objects. Kramer [15] showed that the set of single-cycle class sums

$$\{[(1)^{n-p}(p)]_n ; p = 1, 2, \ldots n\} \tag{19}$$

generates the centre.

In many applications it is useful to start from the centre of a subalgebra and augment it so as to generate the centre of the algebra. Thus, Chen [5] showed that the centre of $CS_n$ can be generated by the set of transposition class sums of the subgroup chain $S_n \supset S_{n-1} \supset \ldots \supset S_2$, i.e.

$$\{[(2)]_k ; k = 2, 3, \ldots, n\}. \tag{20}$$

This result is equivalent to the statement that any member of the centre of $CS_n$ can be expressed as a polynomial over the set of commuting operators consisting of the generators of the centre of $CS_{n-1}$ augmented by the transposition class sum of $CS_n$, $[(2)]_n$.

Katriel and Novoselsky [16] have recently shown that the centre of $CS_n$ can be generated in a similar sense by augmenting the centre of $CS_{n-2}$ with the class sums $[(2)]_n$ and $[(3)]_n$. The derivation of that result, using Partensky's expressions for the eigenvalues of these class sums [10], drew the present author's attention to the role of the contents of the Young shapes, or rather the symmetric polynomials in these, in the representation theory of the symmetric group.

In this section we prove the following theorem.

*Theorem 2.* Any member of the centre of $CS_n$ can be expressed as a polynomial over the set of commuting operators consisting of the generators of the centre of $CS_{n-k}$ augmented by the class sums $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$ of $CS_n$.

To prove this theorem we consider a common eigenfunction $\Phi$ of all the elements of the centre of $CS_{n-k}$ as well as of $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$. We show that $\Phi$ belongs to a well-defined irrep of $S_n$ which is fully specified by the set of eigenvalues corresponding to the generators of $CS_{n-k}$ and the $k$ additional $CS_n$ class sums mentioned above. Since this is valid for any function $\Phi$ with the properties specified above it follows that the generators of $CS_n$ can in fact be expressed in terms of the set consisting of the generators of $CS_{n-k}$ and the first $k$ single-cycle class sums of $CS_n$. The deduction from the properties of the irreps to those of the operators themselves is a special case of the commonly used principle that a property which holds for a complete set of eigenstates is a property of the operator itself.

The function $\Phi$ specified above is a common eigenfunction of the elements of the centre of $CS_{n-k}$ from which fact it follows that it belongs to an irrep $\Gamma_{n-k}$ of $S_{n-k}$. This irrep is fully specified by the eigenvalues of any set of generating class sums.

By lemma 4, the eigenvalues of $[(2)]_{n-k}, [(3)]_{n-k}, \ldots, [(k+1)]_{n-k}$ determine the power sums $\sigma_1(n-k), \sigma_2(n-k), \ldots, \sigma_k(n-k)$. If $k+1>n-k$ (i.e. $k>[\frac{1}{2}(n-1)]$) the set of eigenvalues of $[(2)]_{n-k}, \ldots, [(n-k)]_{n-k}$ determines $\sigma_1(n-k)$, $\sigma_2(n-k), \ldots, \sigma_{n-k-1}(n-k)$ and these power sums themselves determine the higher power sums $\sigma_{n-k}(n-k), \ldots, \sigma_k(n-k)$. Similarly, the eigenvalues of $[(2)]_n$, $[(3)]_n, \ldots$, $[(k+1)]_n$ determine the power sums $\sigma_1(n), \sigma_2(n), \ldots, \sigma_k(n)$. Note that

$$\sigma_l(n)-\sigma_l(n-k)=\sum_{i=1}^{k} \alpha_i^l \qquad l=1,2,\ldots,k \tag{21}$$

where $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a set of contents. By lemma 1, equations (21) determine the set of contents $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ uniquely, so that along with the set of $n-k$ contents corresponding to $\Gamma_{n-k}$ we obtain an additional set of $k$ contents which, together, determine a unique $\Gamma_n$. Since this result holds for any function $\Phi$ specified as above, the theorem follows.

This theorem is useful in the reduction

$$\Gamma_{n_1}\otimes\Gamma_{n_2}=\sum_i \mathscr{C}_i\Gamma^{(i)}_{(n_1+n_2)} \tag{22}$$

which is relevant to the problem of constructing symmetry-adapted multi-cluster wavefunctions [16].

## 5. Specification of irreps with a restricted number of rows

It is well known that irreps of the symmetric group associated with Young shapes consisting of not more than two rows are fully specified by the eigenvalue of the class of transpositions. This property is the origin of the fact that the eigenvalue of the resultant spin operator for systems of identical particles with elementary spins equal to $\frac{1}{2}$ is sufficient to specify the irrep of the symmetric group to which the state of interest belongs.

A generalization of this result, which applies to systems of identical particles with higher elementary spins, is formulated in the following theorem.

*Theorem 3.* If the Young shape of an irrep is known to have at most $k$ rows, the irrep is fully specified by the eigenvalues of the class sums $[(2)]_n, [(3)]_n, \ldots, [(k)]_n$.

To prove the theorem we first note that by lemma 4 the eigenvalues of the class sums $[(2)]_n, [(3)]_n, \ldots, [(k)]_n$ specify the power sums $\sigma_1(n), \sigma_2(n), \ldots, \sigma_{k-1}(n)$.

We consider a partition of $n$ corresponding to an irrep of $S_n$ with at most $k$ rows, i.e. $\lambda_1+\lambda_2+\ldots+\lambda_k=n$ where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$

The power sums corresponding to this irrep are

$$\sigma_r(n)=\sum_{i=1}^{k}\sum_{j=1}^{\lambda_i}(-ij)^r \qquad r=1,2,\ldots,k-1. \tag{23}$$

Using the identities

$$\sum_{j=1}^{\mu} j^r=\sum_{m=0}^{r+1} a_{r,m}\mu^m \qquad a_{r,r+1}\neq 0 \tag{24}$$

equations (23) can be written in the form

$$\sum_{m=0}^{r+1} a_{r,m}p_m=\tilde{\sigma}_r \qquad r=1,2,\ldots,k-1 \tag{25}$$

where

$$p_m=\sum_{i=1}^{k}(\lambda_i-i)^m \tag{26}$$

and

$$\tilde{\sigma}_r=\sigma_r-(-1)^r\sum_{i=1}^{k}\sum_{j=0}^{i-1} j^r.$$

Since $p_0=k$ and $p_1=n-\frac{1}{2}k(k+1)$ we can obtain $p_2$, $p_3$ up to $p_k$ successively, using equation (25). Having obtained these power sums we can obtain the $k$ variables $\lambda_1-1$, $\lambda_2-2, \ldots, \lambda_k-k$. Noting that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$ it follows that $\lambda_1-1>\lambda_2-2>\ldots>\lambda_k-k$ so that the order of the $k$ variables determined using the power sums (26) is unique.

The partition $\lambda_1, \lambda_2, \ldots, \lambda_k$ has thus been uniquely determined, which concludes the proof of theorem 3.

## 6. Some results concerning the minimal number of generators for the centre of the symmetric group algebra

While the full construction of the character table is required if the group-theoretical orthogonality theorem is to be used to reduce a given representation into its irreducible components, the results presented above imply that a rather small subset of class sums is actually needed. Using the approach advocated by Chen [5] we recall that any subspace of functions which is invariant under the group (i.e. carries a representation of the group) can be reduced into the constituent irreps by simultaneous diagonalization of any set of generators of the centre of the group algebra. In fact, any set of operators whose eigenvalues label the irreps uniquely suffices. In view of this last observation it is remarkable that the number of single-cycle classes which are sufficient to provide a complete labelling of the irreps of $S_n$, for any given $n$, is actually much smaller than $n-1$, the number implied by the results presented above. By calculating the eigenvalues of the single-cycle class sums one finds the results presented in table 1 for $n_{\max}(k)$, the maximum value of $n$ for which the set of $k$ class sums $[(2)]_n, [(3)]_n, \ldots, [(k+1)]_n$

Table 1. The number of class sums needed to specify the irreps of various symmetric groups

| Number of class sums used | Order of the maximal symmetric group | Number of irreps in the maximal group |
|---|---|---|
| 1 | 5! | 7 |
| 2 | 14! | 135 |
| 3 | 23! | 1 255 |
| 4 | 41! | 44 583 |

is sufficient to label all the irreps uniquely. Also included in table 1 is the number of irreps (and classes) in the group $S_n$ with $n = n_{max}(k)$.

The rapid increase in the number of irreps involved accounts for the fact that going beyond $k = 4$ becomes computationally prohibitive and probably of little practical use. However, the fact that a complete characterization of the irreps of groups as high as $S_{41}$ is possible with a mere four class sums is remarkable, in view of the fact that even state-of-the-art computer codes for the construction of the character tables of symmetric groups [17] are limited at $n \leq 20$. In fact, an attempt to determine the maximal symmetric group whose irreps are specified by the first five single-cycle class sums was aborted after it was established that for $S_{72}$ each one of the 5392 783 irreps is uniquely labelled in terms of the five corresponding eigenvalues.

## Acknowledgments

Helpful discussions with Professors D Chillag, G James, A Juhasz and I G MacDonald are gratefully acknowledged, as are helpful suggestions by Mr H Katriel. Presentation was considerably improved due to the referees' comments. This research was supported by the Technion VPR Fund–Glasberg-Klein Research Fund, and the Fund for the Promotion of Research at Technion.

## References

[1] de B Robinson G 1961 *Representation Theory of the Symmetric Group* (Toronto University of Toronto Press)
[2] MacDonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford. Clarendon Press)
[3] James G and Kerber A 1981 *The Representation Theory of the Symmetric Group* (Reading, MA Addison-Wesley)
[4] Coleman A J 1968 *Advances in Quantum Chemistry* vol 4 ed P-O Lowdin (New York Academic Press)
[5] Chen J-Q 1989 *Group Representation Theory for Physicists* (Singapore World Scientific)
[6] Katriel J 1989 *Int. J. Quantum Chem* **35** 461
[7] Katriel J 1991 *Int J Quantum Chem* **39** 593
[8] Katriel J *Israel J Chem* in press
[9] de B Robinson G and Thrall R M 1953/1954 *Michigan Math J* **2** 157
[10] Partensky A 1972 *J. Math Phys* **13** 1503
[11] Jucys A A 1974 *Rep Math Phys* **5** 107
[12] Suzuki M 1987 *Proc. Symp Pure Math.* (*Am Math. Soc*) **47** 317
[13] Pauncz R and Katriel J *Int. J. Quantum Chem.* in press
[14] Farahat H K and Higman G 1959 *Proc. R. Soc London* **250** 212
[15] Kramer P 1966 *Z Naturforsch* **21a** 657
[16] Katriel J and Novoselsky A *Ann Phys, NY* in press
[17] Liu X and Balasubramanian K 1989 *J Comput. Chem* **10** 417